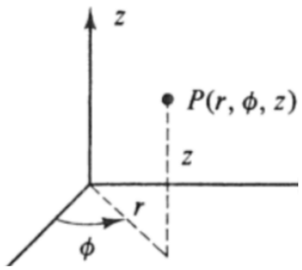
ELEN E3401 | Solution
Electromagnetics | **Problem Set 4** | March 4, 2025

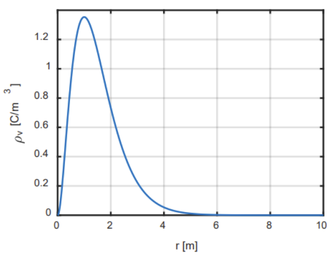
Problem 1: Charge Density and D in Cylindrical Coordinates [+40 pts]

In a certain region of space, the charge density is given in cylindrical coordinates by the function:

$$\rho_v(r, \phi, z) = 10r^2 e^{-2r} \quad [\mathbf{C/m^3}] \tag{1}$$

We start with the integral form of Gauss's Law for the electric flux density D:





$$\oint \vec{D} \cdot d\vec{s} = Q. \tag{2}$$

We can evaluate both sides independently and then recombine at the end. First, we can solve for the left side of the equation by enclosing the charge density ρ_v in a cylindrical Guassian surface: $d\vec{s} = \hat{r} r \, d\phi \, dz$. This can be done by the following:

$$\begin{aligned}
\oint \vec{D} \cdot d\vec{s} &= \iint |D|\hat{r} \cdot (\hat{r} r \, d\phi \, dz), \\
&= |D| r \int_0^{2\pi} d\phi \int_0^L dz, \\
&= |D| r (2\pi)(L) = |D|(2\pi r L).
\end{aligned} \tag{3}$$

Now that the left side has been solved, we can move on to solving for the enclosed charge Q. Again, we will use the same Guassian surface to enclose the volumetric charge density ρ_v:

$$\begin{aligned}
Q &= \int_{\mathcal{V}} \rho_v \, d\mathcal{V}, \\
&= \int_0^{r'} \rho_v r \, dr \int_0^{2\pi} d\phi \int_0^L dz, \\
&= 20\pi L \int_0^{r'} \left(r^2 e^{-2r}\right) r \, dr = 20\pi L \int_0^{r'} r^3 e^{-2r} \, dr.
\end{aligned} \tag{4}$$

Note: I use r' here to show that there is a difference in the integration variable and the value r'. However, I will drop this moving forward to minimize complexity of the integral. This integral requires several iterations of integration-by-parts:

$$\begin{aligned}
Q &= 20\pi L\left(\left.\frac{-r^3e^{-2r}}{2}\right|_0^r + \int_0^r \frac{3r^2e^{-2r}}{2}dr\right), \\
&= 20\pi L\left(\left.\frac{-r^3e^{-2r}}{2}\right|_0^r + \left.\frac{-3r^2e^{-2r}}{4}\right|_0^r + \int_0^r \frac{3re^{-2r}}{2}dr\right), \\
&= 20\pi L\left(\left.\frac{-r^3e^{-2r}}{2}\right|_0^r + \left.\frac{-3r^2e^{-2r}}{4}\right|_0^r + \left.\frac{-3re^{-2r}}{4}\right|_0^r + \int_0^r \frac{3e^{-2r}}{4}dr\right), \\
&= 20\pi L\left(\left.\frac{-r^3e^{-2r}}{2}\right|_0^r + \left.\frac{-3r^2e^{-2r}}{4}\right|_0^r + \left.\frac{-3re^{-2r}}{4}\right|_0^r + \left.\frac{-3e^{-2r}}{8}\right|_0^r\right).
\end{aligned}$$

Now, we can evaluate each term.

$$\begin{aligned}
Q &= 20\pi L\left(\frac{-r^3}{2}e^{-2r} + \frac{-3r^2}{4}e^{-2r} + \frac{-3r}{4}e^{-2r} + \frac{-3}{8}e^{-2r} + \frac{3}{8}\right), \\
&= 20\pi L\left(\frac{3}{8} - \frac{(4r^3+6r^2+6r+3)e^{-2r}}{8}\right). \qquad (5)
\end{aligned}$$

We have both sides of Gauss's Law solved for and can substitute them back in.

$$\begin{aligned}
|D|(2\pi rL) &= 20\pi L\left(\frac{3}{8} - \frac{(4r^3+6r^2+6r+3)e^{-2r}}{8}\right), \\
|D| &= \frac{10}{r}\left(\frac{3}{8} - \frac{(4r^3+6r^2+6r+3)e^{-2r}}{8}\right). \qquad (6)
\end{aligned}$$

And expressing the electric flux density as vector where $\vec{D} = |D|\,\hat{r}$, we arrive at the solution:

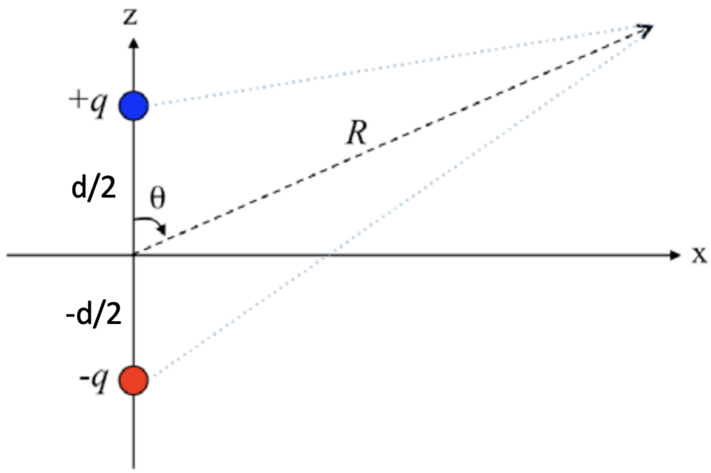
$$\vec{D} = \frac{10}{r}\left(\frac{3}{8} - \frac{(4r^3+6r^2+6r+3)e^{-2r}}{8}\right)\hat{r} \quad [\mathbf{C/m^2}]. \qquad (7)$$

Problem 2: Electric Dipole [+30 pts]

Two equal and opposite point charges, q and $-q$, are located at $(0,0,\frac{d}{2})$ and $(0,0,-\frac{d}{2})$ respectively, as shown in the figure below. This arrangement is known as an *electric dipole*. Use first-order Taylor expansions to show that the total electric field due to this electric dipole at very large distances from the origin as compared to the dipole separation, d, is given approximately by:

$$E \approx \frac{qd}{4\pi\epsilon_0 R^3}(2\cos(\theta)\hat{R} + \sin(\theta)\hat{\theta})$$

First, consider the individual contributions of each charge at the point R. Define new vectors $\overrightarrow{R_+}$ and $\overrightarrow{R_-}$ for the positive and negative charges respectively.



$$\begin{aligned}\overrightarrow{R}_+ &= R\hat{R} - \frac{d}{2}\hat{z} \\ \overrightarrow{R}_- &= R\hat{R} + \frac{d}{2}\hat{z}\end{aligned} \tag{8}$$

The total field is the superposition of the fields due to +q and -q:

$$\begin{aligned}\overrightarrow{E} = \overrightarrow{E}_+ + \overrightarrow{E}_- &= \frac{q}{4\pi\epsilon_0}\frac{\overrightarrow{R}_+}{|\overrightarrow{R}_+|^3} - \frac{q}{4\pi\epsilon_0}\frac{\overrightarrow{R_-}}{|\overrightarrow{R_-}|^3} \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{\overrightarrow{R}_+}{|\overrightarrow{R}_+|^3} - \frac{\overrightarrow{R}_-}{|\overrightarrow{R}_-|^3}\right)\end{aligned} \tag{9}$$

Now we can evaluate the magnitudes $|\overrightarrow{R}_+|$ and $|\overrightarrow{R}_-|$ so we can rewrite the expression:

$$\begin{aligned}|\overrightarrow{R_+}| = |R\hat{R} - \frac{d}{2}\hat{z}| &= \sqrt{R^2 + \left(\frac{d}{2}\right)^2 - 2R\frac{d}{2}\hat{R}\cdot\hat{z}} \\ &= \sqrt{R^2 - Rd\hat{R}\cdot\hat{z}} \quad \text{Using the approximation } R >> d \\ &= R\sqrt{1 - \frac{d}{R}\hat{R}\cdot\hat{z}}\end{aligned} \tag{10}$$

Now using the Taylor series first-order expansion

$$\sqrt{1-x} \approx 1 - \frac{x}{2} \quad \text{for } x << 1 \tag{11}$$

$$|\overrightarrow{R}_+| = R(1 - \frac{d}{2R}\hat{R}\cdot\hat{z}) = R\left(1 - \frac{d}{2R}\cos(\theta)\right) \tag{12}$$

Similarly

$$|\overrightarrow{R}_-| = R(1 + \frac{d}{2R}\hat{R}\cdot\hat{z}) = R\left(1 + \frac{d}{2R}\cos(\theta)\right) \tag{13}$$

Using the Taylor series expansion for

$$(\frac{1}{1+x})^3 \approx 1 - 3x$$

$$\begin{aligned}\frac{\overrightarrow{R}_+}{|\overrightarrow{R}_+|^3} - \frac{\overrightarrow{R}_-}{|\overrightarrow{R}_-|^3} &\approx \frac{1}{R^3}\left((R\hat{R} - \frac{d}{2}\hat{z})(1 + \frac{3d}{2R}\cos(\theta)) - (R\hat{R} + \frac{d}{2}\hat{z})(1 - \frac{3d}{2R}\cos(\theta))\right) \\ &\approx \frac{1}{R^3}\left(\hat{R}\left(R + \frac{3d}{2}\cos(\theta) - R + \frac{3d}{2}\cos(\theta)\right) + \hat{z}(-d)\right) \\ &\approx \frac{1}{R^3}\left(3d\cos(\theta)\hat{R} - d\hat{z}\right)\end{aligned} \tag{14}$$

Thus, the full expression for the field is:

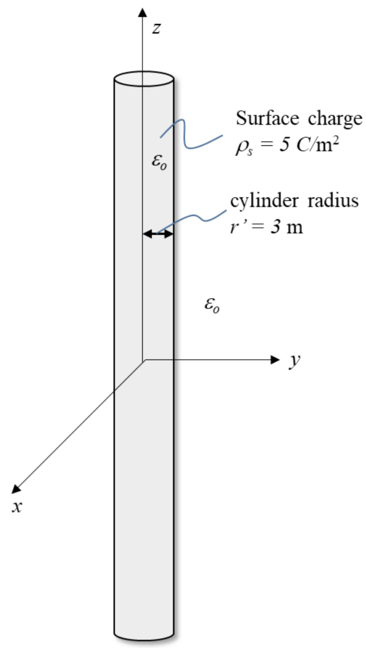
$$\overrightarrow{E} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R^3}\left(3d\cos(\theta)\hat{R} - d\hat{z}\right)\right) = \frac{qd}{4\pi\epsilon_0 R^3}\left(3\cos(\theta)\hat{R} - \hat{z}\right) \tag{15}$$

But we know that $\hat{z} = \cos(\theta)\hat{R} - \sin(\theta)\hat{\theta}$, allowing us to further manipulate to complete radial coordinates and to match the equation we are trying to derive.

$$\overrightarrow{E}_{\text{dipole}}(R,\theta) = \frac{qd}{4\pi\epsilon_0 R^3}\left(2\cos(\theta)\hat{R} + \sin(\theta)\hat{\theta}\right) \tag{16}$$

Problem 3: Charge Distribution Electric Field [+30 pts]

A very long, thin cylindrical shell of radius $r' = 3$ meters in free space has a surface charge density equal to +5 C/m^2 ($\rho_s = +5\frac{C}{m^2}$), as shown in the figure below:



a **Obtain the electric flux density ($\overrightarrow{D}$) (magnitude and direction) inside the cylinder (for $r' < 3m$). Plot and label the electrical field, E, for this region. (2pts)**

Inside the cylinder, we choose our Gaussian surface to be a cylinder with $r < r'$ such that all field components are either parallel or perpendicular to the chosen surface. Recall the integral form of Gauss's Law:

$$\oint \overrightarrow{E} \cdot d\overrightarrow{s} = \frac{Q_{\text{enclosed}}}{\epsilon_0} \tag{17}$$

Where here the amount of charge enclosed by our surface is 0, so

$$\oint \overrightarrow{E} \cdot d\overrightarrow{s} = 0 \tag{18}$$

And we also know $\overrightarrow{E} \cdot d\overrightarrow{A} = 0$ for the top and bottom caps of the cylinder, since they are perpendicular. Calculating the flux through the sides of the cylinder:

$$\int \overrightarrow{E} \cdot d\overrightarrow{s} = |\overrightarrow{E}| \int ds = |\overrightarrow{E}| r 2\pi l = \frac{Q_{\text{enclosed}}}{\epsilon_0} = 0 \tag{19}$$

Therefore we have $\overrightarrow{E} = 0$ inside the cylinder and since we are in free space, the constitutive relation tells us

$$\overrightarrow{D} = \epsilon_0 \overrightarrow{E} = 0 \Longrightarrow \overrightarrow{D} = 0 \tag{20}$$

Plotting $\overrightarrow{E}$ in this region, in Fig. 1 below, it is clear that the overlapping field lines (represented by the arrows) will cancel out by superposition inside the cylinder

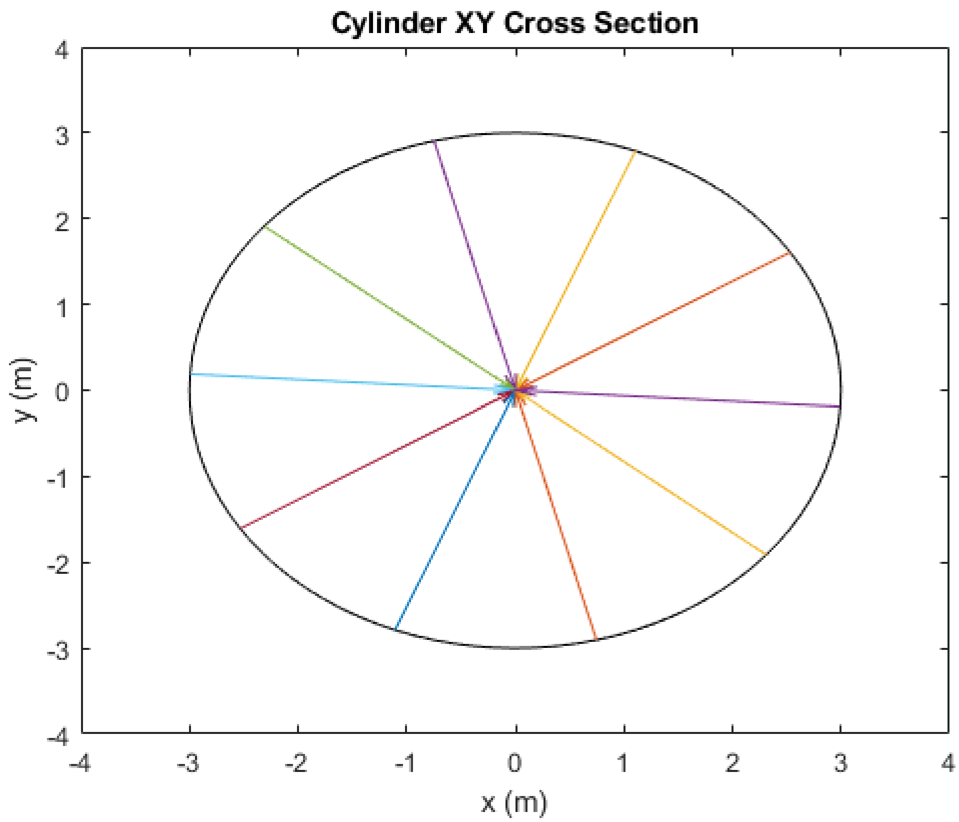


Figure 1: The $\overrightarrow{E}$ inside the cylinder. For every $\overrightarrow{E}$ line pointing towards the center, there is an equal and opposite field line to cancel it out.

b **Obtain the electric flux density ($\overrightarrow{D}$) (magnitude and direction) <u>outside</u> the cylinder (for $r' > 3m$). Plot and label the electrical field, E, for this region. (2pts)**

Outside, we choose our Gaussian surface to be a larger cylinder with radius $r > r'$. Again, the field lines (radial) are parallel to the caps, so we only need to calculate the flux through the side wall where the field lines are perpendicular to the Gaussian surface.

$$\int \overrightarrow{E} \cdot d\overrightarrow{s} = \frac{Q_{\text{enclosed}}}{\epsilon_0} \tag{21}$$

Where Q_{enclosed} is now the total charge on the cylindrical shell (i.e. the surface charge density times the surface area) and thus

$$Q_{\text{enclosed}} = (\rho_s)(2\pi r' l) = 10\pi r' l \quad \text{C} \tag{22}$$

So our equation becomes

$$\int \vec{E} \cdot d\vec{s} = \frac{10\pi r' l}{\epsilon_0} \tag{23}$$

$$|E| \int ds = \frac{10\pi r' l}{\epsilon_0} \quad (\vec{E} \perp d\vec{s} \text{everywhere for walls}) \tag{24}$$

$$|E|(2\pi l r) = \frac{10\pi r' l}{\epsilon_0} \tag{25}$$

$$|E| = \frac{5}{\epsilon_0}\left(\frac{r'}{r}\right) \tag{26}$$

$$\vec{E} = |E|\hat{r} = \frac{5}{\epsilon_0}\left(\frac{r'}{r}\right)\hat{r} \tag{27}$$

$$\vec{D} = 5\left(\frac{r'}{r}\right)\hat{r} \tag{28}$$

We plot $\vec{E}$ in this region, in Fig. 2 below. As the fields separate, the magnitude of $\vec{E}$ decreases.

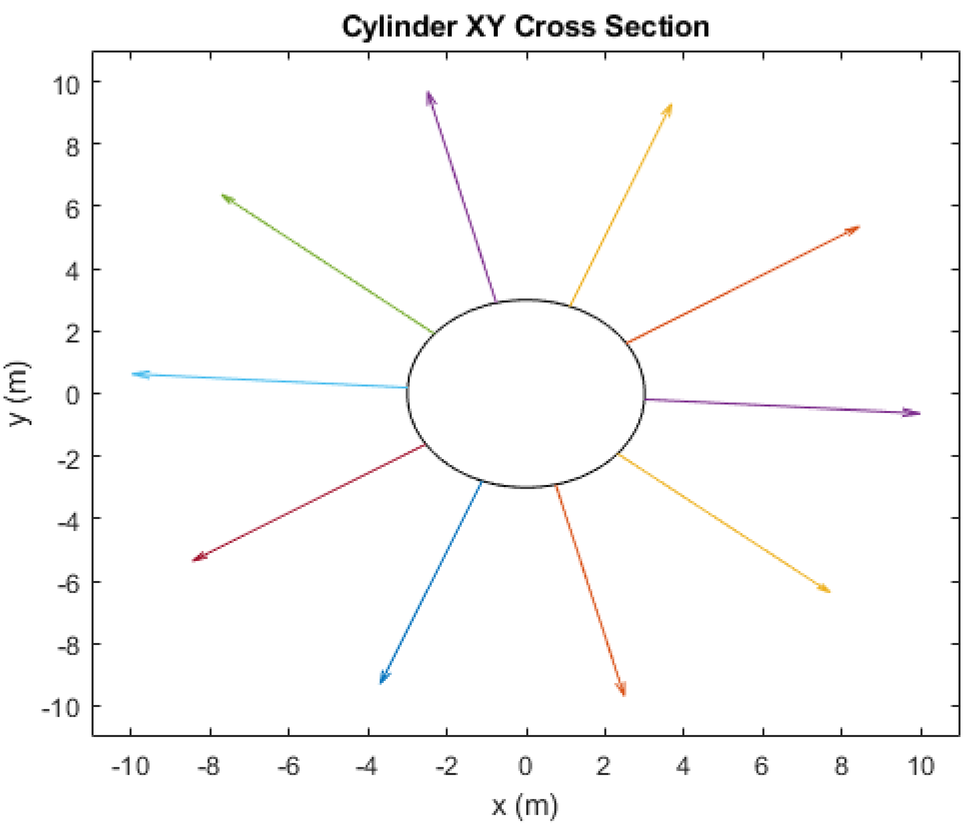


Figure 2: The $\overrightarrow{E}$ outside the cylinder.